Critical gravitational collapse with angular momentum: from critical exponents to scaling functions

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We investigate the threshold of gravitational collapse with angular momentum, under the assumption that the critical solution is spherical and self-similar and has two growing modes, namely one spherical mode and one axial dipole mode (threefold degenerate). This assumption holds for perfect fluid matter with the equation of state $p = \kappa\rho$ if the constant κ is in the range $0 < \kappa < 1/9$. There is a region in the space of initial data where the mass and angular momentum of the black hole created in the collapse are given in terms of the initial data by two universal critical exponents and two universal functions of one argument. These expressions are similar to those for the correlation length and the magnetization in a ferromagnet near its critical point, as a function of the temperature and the external magnetic field. We discuss qualitative features of the scaling functions, and hence of critical collapse with high angular momentum.

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Contents

I. INTRODUCTION

An isolated system in general relativity ends up in one of three final states. It either collapses to a black hole, forms a star, or disperses completely. The phase space of isolated gravitating systems is therefore divided into basins of attraction. One cannot usually tell into which basin of attraction a given data set belongs by any other method than evolving it in time to see what its final state is. The study of these boundaries in phase space, in particular of the boundary between black hole formation and dispersion, is the subject of the new field of critical collapse [1,2].

The pioneering work of Choptuik [3] has shown that the black hole threshold is both richer in structure and simpler than naively expected. Choptuik carried out systematic high precision numerical collapse simulations in the toy model of a spherically symmetric massless scalar field coupled to general relativity. He explored the black hole threshold

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by means of smooth one-parameter families of initial data. A generic family contains both collapsing and dispersing data sets. Choptuik found that locally each family crosses the black hole threshold only once. This suggests that the collapse threshold is a smooth hypersurface in phase space.

The black hole mass shows a universal power-law scaling as a function of distance from the black hole threshold, as measured by the parameter p of the one-parameter family of initial data. Let a black hole be formed for $p > p_*$, and let dispersion occur for $p < p_*$. The critical value p_* of the parameter p depends on the family, and can only be determined through a bisection search. Choptuik found that the black hole mass is approximately

$$M(p) \simeq C(p - p_*)^\gamma \quad \text{for } p > p_*, \tag{1}$$

where the critical exponent γ is independent of the family (universal), with a numerical value of $\gamma \simeq 0.374$ for the scalar field.

Furthermore, all evolutions with $p \simeq p_*$, on either side of p_*, and for any family, pass through an intermediate attractor, before finally dispersing or forming a black hole. This "critical solution" is self-similar. In the case of the scalar field, the self-similarity is a discrete symmetry. In the case of the perfect fluid, it is continuous. By virtue of being an intermediate attractor, the critical solution has precisely one growing mode. The critical exponent for the black hole mass can be calculated from this growing mode.

Subsequent work has found similar critical phenomena in many other matter models (see [2] for a review). Unfortunately, most work so far has been limited to a spherically symmetric, uncharged situation in which a Schwarzschild black hole is formed. Generic black holes, however, have angular momentum L and electric charge Q as well as mass M, and we do not yet know what happens if one fine-tunes to the black hole threshold along a one-parameter family of data with significant angular momentum and/or charge.

Today, the role of angular momentum and charge in critical collapse is understood only in the limit where they are so small that they can be treated as linear perturbations of a spherically symmetric and uncharged scenario throughout the collapse, from the initial data to the final black hole. One can then make a connection from a small perturbation in the initial data representing charge or angular momentum to a linear perturbation of the final Schwarzschild black hole that takes it into a Kerr-Newman black hole.

Specifically, critical collapse with a small amount of electric charge has been investigated in the model of a spherically symmetric complex scalar field coupled to a Maxwell field. A critical exponent for the black hole charge Q has been calculated by keeping track of the least slowly decaying charged perturbation of the critical solution [4]. This prediction has subsequently been verified in collapse simulations [5].

Similarly, critical collapse with a small amount of angular momentum has been investigated in the model of a perfect fluid with equation of state $p = \kappa\rho$, where p is the pressure, ρ the total energy density, and κ is a constant with $0 < \kappa < 1$. In exact spherical symmetry, the critical exponent γ for the black hole mass is again independent of the initial data but depends on κ [6–9]. Perturbing around spherical symmetry, a critical exponent for black hole angular momentum L was calculated from the least slowly decaying axial dipole perturbation in [10]. The numerical value of the critical exponent has been corrected in [11]. For the scalar field, a critical exponent for L was calculated in second order perturbation theory around spherical symmetry [12]. Both these predictions still await testing, as high-precision simulations of rotating collapse to a black hole are not yet available.

In this paper we go beyond the assumption that angular momentum is a small perturbation throughout. This is necessary because the corrected results for the axial dipole perturbations of the critical fluid in [11] show that all such perturbations decay only for the equations of state $k > 1/9$. For $k < 1/9$, there is actually a single growing mode. This means that in the limit in which initial data with a small amount of angular momentum are fine-tuned to the black hole threshold, there are two competing growing modes of the critical solution, one spherical and one to do with differential rotation. The latter is threefold degenerate because rotation is an axial vector. Which of the two modes first reaches nonlinearity depends on their initial amplitude and growth rate. In particular, if the initial data are sufficiently close to the collapse threshold, rotation may dominate the late stages of the evolution even if it was only a small perturbation in the initial data, and the final black hole could be very small but rapidly rotating.

We begin our presentation in section II by reviewing the qualitative picture of critical collapse in terms of a dynamical system, and establishing notation. In section III we identify an intermediate attractor through which solutions near the black hole threshold are funnelled. In section IV we discuss the initial data that reach this attractor, and in V we follow them from the intermediate attractor to the black hole end state. We find that the black hole mass and angular momentum depend on the initial data only through universal functions of one combination of the initial data parameters. In this paper, we do not calculate these "universal scaling functions", but their mere existence substantially constrains the phenomenology of critical collapse resulting in small but rapidly spinning black holes. We discuss this phenomenology in section VII. The present work has been motivated by the close analogy between critical collapse and critical phase transitions in statistical mechanics, and represents an extension of that analogy. This analogy is discussed separately in section VIII. We summarize in section IX.

In talking about black holes, it is customary to refer not to its angular momentum L but to its specific angular momentum $a \equiv L/M$, and we adopt this convention here.

II. THE DYNAMICAL SYSTEMS PICTURE

The time evolution of general relativity can be considered as an (infinite-dimensional) dynamical system once the coordinate freedom of general relativity has been fixed by imposing suitable coordinate conditions on parts of the metric. Here we review this phase space picture in preparation for the calculation of critical exponents and scaling functions in the next section. Black holes on the one side, and flat spacetime on the other, are attracting fixed points. Their basins of attraction are separated by the black hole threshold, or critical surface. It is obviously a hypersurface of codimension one. The numerical evidence is consistent with the assumption that it is smooth, and not, for example, fractal. By definition, the critical surface is a closed dynamical system in its own right. Within the full phase space, it is a repeller. Its attracting fixed points or limit cycles are then attractors of codimension one in the full phase space, with exactly one growing perturbation mode. They are called critical fixed points in dynamical systems language, or critical solutions in spacetime language.

Any trajectory beginning near the critical surface, but not necessarily near the critical point, moves almost parallel to the critical surface towards the critical point. As the critical point is approached, the parallel movement slows down, and the phase point spends some time near the critical point. Then the phase space point moves away from the critical point in the direction of the growing mode, and finally ends up on a stable fixed point. This is the origin of universality: any initial data set that is close to the black hole threshold (on either side) evolves to a spacetime that approximates the critical spacetime for some time. When it finally approaches either empty space or a black hole it does so on a trajectory that appears to be coming from the critical point itself.

Not surprisingly, attractors of this dynamical system have higher symmetry, when considered as Cauchy data for a spacetime, than generic points in the phase space. The critical solutions found at the black hole threshold, at least those known to date, are either static, periodic, continuously self-similar, or discretely self-similar. Here we consider only the continuously self-similar critical solutions, which give rise to scaling laws of the form (1). The model we focus on here, the perfect fluid with equation of state $p = \kappa\rho$, has this type of critical solution. A spacetime is continuously self-similar, or homothetic, if there exists a vector field χ such that the Lie derivative of the spacetime metric along χ obeys

$$\mathcal{L}_\chi g_{ab} = -2g_{ab}. \tag{2}$$

χ is called a homothetic vector field. It is a special case of a conformal Killing vector field. By the Einstein equations, continuous self-similarity implies that the matter stress-energy obeys $\mathcal{L}_\chi T_{ab} = 0$.

In general relativity, each data set locally determines a unique spacetime. But a given spacetime can be broken up into a sequence of data sets, that is, foliated by spacelike hypersurfaces, in an infinite number of ways, even if the initial hypersurface is kept fixed. This means that the same physical spacetime can be described by an infinite number of curves through the phase space of general relativity. Turning general relativity into a dynamical system therefore requires some prescription that returns a lapse and shift for each point in phase space. In the following, let τ be the time coordinate of the dynamical system, and let x be the three spatial coordinates.

Calculations in critical phenomena require a prescription in which the self-similar spacetimes are fixed points, and discretely self-similar spacetimes are limit cycles. This means that for those sets of Cauchy data that will evolve into a self-similar spacetime, the prescription returns a lapse α and shift vector β^a such that the time vector $\partial/\partial\tau$ is the homothetic vector χ:

$$\left(\frac{\partial}{\partial\tau}\right)^a \equiv \alpha n^a + \beta^a = \chi^a, \tag{3}$$

where n^a is the unit normal vector on the constant τ slices. Certain possible prescriptions, such as "K-freezing lapse" with "minimal distortion shift" have been identified [13], and we now assume that one such choice has been made. Note that while $\partial/\partial\tau$ becomes spacelike at large distances from the center of collapse, this does not preclude surfaces of constant τ from being everywhere spacelike. The spacetime metric in these coordinates is

$$g_{\mu\nu}(\tau, x) = l^2 e^{-2\tau} \bar{g}_{\mu\nu}(x), \tag{4}$$

where $\bar{g}_{\mu\nu}$ does not depend on τ, and where l is an arbitrary fixed length scale. Foliating by surfaces of constant τ we obtain an induced metric and intrinsic curvature of the form

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$$g_{ij}(\tau, x) = l^2 e^{-2\tau} \bar{g}_{ij}(x), \qquad K_{ij}(\tau, x) = l e^{-\tau} \bar{K}_{ij}(x). \tag{5}$$

We now turn this around to define the rescaled dynamical variables

$$\bar{g}_{ij}(\tau, x) \equiv l^{-2} e^{2\tau} g_{ij}(\tau, x,), \qquad \bar{K}_{ij}(\tau, x) \equiv l^{-1} e^{-\tau} K_{ij}(\tau, x) \tag{6}$$

for the metric, and similar rescaled variables for the matter fields. For a perfect fluid with 4-velocity u^μ and comoving energy density ρ, for example, we define

$$\bar{u}^i \equiv l e^{-\tau} u^i, \qquad \bar{\rho} \equiv l^2 e^{-2\tau} \rho. \tag{7}$$

In the variables $Z \equiv \{\bar{g}_{ij}, \bar{K}_{ij}, \bar{u}^i, \bar{\rho}, \ldots\}$, and with a suitable choice of lapse and shift, a CSS solution will take the form $Z(x, \tau) = Z_*(x)$. Full physical initial data consist of $Z(x, \tau_0)$ and the value of τ_0 itself, which provides an overall scale and allows one to reconstruct the physical variables from the barred variables.

It is helpful to think of the coordinates x and τ and the barred variables as dimensionless, of $l e^{-\tau}$ as having a dimension of length (or time, or mass, in units $c = G = 1$), and of the unbarred quantities as having their natural dimensions, with $g_{\mu\nu}$ having a dimension of l^2 because the coordinates are dimensionless. The Einstein and matter equations in the barred variables are much like the usual ones, but are dimensionless. $e^{-\tau}$ only appears in the combination $l e^{-\tau}$. In vacuum gravity, with a minimally coupled massless scalar field, or with a perfect fluid with equation of state $p = \kappa \rho$ fluid, no dimensionful constants appear in the field equations. Therefore there are no constants to cancel l, and so no powers of $e^{-\tau}$ can appear explicitly in the dimensionless equations in the barred variables. This means that solutions are possible which do not depend on τ, and so are self-similar. If dimensionful constants of dimension l^{-n}, with $n > 0$, appear in the field equations (for example, a mass term in the scalar wave equation), they are multiplied by $l^n e^{-n\tau}$ in the scale-invariant equations, and so become dynamically irrelevant as $\tau \to \infty$, which corresponds to very small scales. This allows for solutions that become asymptotically self-similar on small scales (large τ).

III. THE INTERMEDIATE LINEAR REGIME

For what follows it is helpful to reformulate the mass-scaling law in spherical symmetry, Eq. (1), by absorbing the family-dependent constants C and p_* into a new parameter $\bar{p}$ that is a linear function of the parameter p, so that the law is now

$$M \simeq \bar{p}^\gamma \quad \text{for } \bar{p} > 0 \tag{8}$$

for every 1-parameter family of initial data. It is an experimental fact that this is possible, with each data set assigned only one value of $\bar{p}$, independently of the 1-parameter family of which it is considered a part. This means that $\bar{p}$ is a scalar function on the infinite-dimensional phase space, independently of any 1-parameter families. M is also a scalar, but it is not regular at the black hole threshold, while $\bar{p}$ is regular with non-vanishing gradient. $\bar{p}$ is therefore a good coordinate on phase space in a neighborhood of the black hole threshold. This clarifies the invariant meaning of the critical exponent γ: it is the unique power that relates the observed scalar M on phase space to another scalar $\bar{p}$ that is a good coordinate.

Based on the phase space picture, the critical exponent γ for the black hole mass can be calculated in a manner suggested by Evans and Coleman [6] and spelled out by Koike, Hara and Adachi [14] and Maison [8]. Here we repeat this calculation, but now including the effects of angular momentum in the initial data. We do this in the abstract notation Z which applies not only to the perfect fluid with $k < 1/9$, but equally to any other model that has a spherically symmetric, continuously self-similar solution with precisely two growing perturbation modes. One of these must be spherically symmetric, and the other one must have an axial dipole ($l = 1$) angular dependence. All other spherical and nonspherical perturbation modes (there are infinitely many) must decay. The generalization from a continuously self-similar to a discretely self-similar critical solution is trivial.

As the background solution we consider is continuously self-similar, with $Z(x, \tau) = Z_*(x)$, the perturbations of this background depend on τ only exponentially. We can write a generic perturbation on the background $Z_*(x)$ as a sum over perturbation modes of the form $e^{\lambda_i \tau} Z_i(x, \Omega)$, where the index i labels the perturbation spectrum. The two growing modes in particular are designated $Z_0(x)$ (for the spherically symmetric one) and $\vec{Z}_1(x, \Omega)$ (for the axial dipole mode). Here the arrow indicates that $\vec{Z}_1$ is 3-fold degenerate, for $m = -1, 0, 1$. Ω is shorthand for the angular dependence on the coordinates (θ, φ). Perturbing around spherical symmetry, to linear order rotation is associated with the axial dipole mode, and in particular, the angular momentum of the spacetime is proportional to the r^{-2}

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falloff of the dipole mode at spacelike infinity. This is true for regular perfect fluid spacetimes, but also for black holes: outside the horizon, a Kerr black hole with $a \ll M$ can be written as an axial dipole linear perturbation of a Schwarzschild black hole. The significance of the axial dipole perturbation is discussed in more detail in Refs. [12] and [15]. As a matter of notation, we go from the basis $m = -1, 0, 1$ to the basis xyz of $l = 1$ harmonics, so that $\vec{Z}_1$ is an axial vector in 3-space that points in the direction of angular velocity.

In this paper, we consider solutions that go through an intermediate time regime in which they are well approximated by a self-similar, spherically symmetric solution plus small perturbations, the "intermediate linear regime". Furthermore, in this regime, we neglect the decaying linear perturbation modes, so that we have the approximation

$$Z(x, \tau, \Omega) \simeq Z_*(x) + A e^{\lambda_0 \tau} Z_0(x) + \vec{B} \cdot e^{\lambda_1 \tau} \vec{Z}_1(x, \Omega) + \text{decaying modes.} \tag{9}$$

In the following we show that all solutions in a ball straddling the hole threshold are in fact funneled through the intermediate linear regime. Working backwards, in section IV we construct families of non-spherically symmetric data that go through the intermediate linear regime. Working forwards, in section V, we extract Cauchy data in the intermediate linear regime, and predict the mass and specific angular momentum of the black hole from these intermediate data.

IV. INITIAL DATA WITH ROTATION

We now construct generic families of initial data that reach the intermediate linear regime (9). To do this, we shall first introduce a parameter controlling angular momentum, and then add a second, generic, parameter which allows us to tune to the black hole threshold.

Consider first infinitesimally small deviations from a spherically symmetric data set that does form a black hole. (For the moment we do not assume that we are at the black hole threshold.) Perturbations with different multipoles, and axial and polar perturbations, then remain decoupled throughout the evolution, and so the angular momentum of the final black hole is linearly related to the axial dipole part of the initial data. However, while the specific angular momentum of the black hole is a vector $\vec{a}$ (three numbers), the axial dipole part of the initial data for a perfect fluid consists of a vector of functions $\vec{\beta}(r)$, which characterize differential rotation. We can explore this 3∞-dimensional space by means of many different 3-parameter families of initial data, with the vector-valued parameter $\vec{q}$ chosen so that $\vec{q} = 0$ corresponds to spherical symmetry, and so that $\vec{q} \to -\vec{q}$ corresponds to $\vec{\beta}(r) \to -\vec{\beta}(r)$ for all r. Clearly, this is a sufficient (but by no means a necessary) criterium for the relation $\vec{a}(-\vec{q}) = -\vec{a}(\vec{q})$ in the final black hole. It is also clear that we have $M(-\vec{q}) = M(\vec{q})$ for the black hole mass. (In fact, from this it follows that to linear order in perturbation theory, M does not depend on $\vec{q}$.

But we need not stop at perturbations of spherical symmetry. In order to make Schwarzschild black holes, one does not need spherically symmetric initial data. Note that a Schwarzschild black hole has three commuting reflection symmetries (also called octant symmetry), while a Kerr black hole has only one reflection symmetry (through the equatorial plane). Any reflection symmetry in the initial data is maintained during evolution. Octant symmetry in the initial data is therefore a sufficient (but not a necessary) criterium for producing a non-rotating black hole. This leads us to consider 3-parameter families of initial data that are not a perturbation of spherical symmetry: we construct them so that the data with $\vec{q} = 0$ are octant-symmetric, and that data with $\vec{q} \neq 0$ contain a point such that $\vec{q} \to -\vec{q}$ corresponds to a reflection through that point. This symmetry between $\vec{q}$ and $-\vec{q}$ is conserved by the time evolution, and so again we have the relations

$$\vec{a}(-\vec{q}) = -\vec{a}(\vec{q}), \qquad M(-\vec{q}) = M(\vec{q}). \tag{10}$$

While octant symmetry of the initial data is a sufficient criterium for the absence of angular momentum, it seems unlikely that one can give a *necessary* criterium for the final black hole to have zero angular momentum. But we can turn the argument around and formally *define* the parameter $\vec{q}$ to have the properties (10). Such a definition may seem circular, but it is in fact the analogue of the definition of the scalar function $\bar{p}$, in that from the family-dependent parameter $\vec{q}$ we can define a parameter $\bar{\vec{q}}$ that is a scalar on the space of initial data and which is related in a simple universal way to the black hole specific angular momentum $\vec{a}$.

We have defined the vector parameter $\vec{q}$ without assuming that we were close to the black hole threshold. However, if the black hole threshold is a surface of codimension one in the full space of non-spherical initial data, then fine-tuning any other parameter p in the initial data should take us to the black hole threshold, for any fixed value of $\vec{q}$. With this motivation, consider now 4-parameter families of initial data with the following properties:

1. The initial data depend analytically on p and $\vec{q}$ in a neighborhood of the black hole threshold.
2. If for any p and $\vec{q}$ a black hole is formed, its specific angular momentum and mass obey (10) for any p.

3. For $\vec{q} = 0$, there is a p_* such that a black hole forms if and only if $p > p_*$.

We should note that p_* is the value of p at the threshold only for $\vec{q} = 0$. The black hole threshold is locally described by $p = p_{\mathrm{crit}}(\vec{q})$ with $p_{\mathrm{crit}}(0) = p_*$.

Do data sets from such a family go through the intermediate linear regime (9)?

a) If $\vec{q} = 0$ corresponds to spherical symmetry, and if $|\vec{q}|$ is sufficiently small, the answer is yes. In this case, the deviations from spherical symmetry are described by perturbation theory throughout. We can then appeal to the experimental fact that in spherical symmetry, the basin of attraction of the spherically symmetric critical solution is the entire black hole threshold, including data that are far from the critical solution.

b) If $\vec{q}$ does not correspond to spherical symmetry, the answer is unknown. For example, there is no guarantee that the spherically symmetric critical solution is reached from near-critical data that are octant-symmetric (and therefore non-rotating) but highly oblate or prolate. But it is suggested by the fact that a spherical critical solution exists, and that critical solutions apparently have as much symmetry as possible. (In critical collapse in pure gravity, the critical solution cannot be spherically symmetric, but it is axisymmetric).

V. UNIVERSAL SCALING FUNCTIONS

We are now ready to calculate $\vec{a}(p, \vec{q})$ and $M(p, \vec{q})$. The amplitudes of Z_0 and $\vec{Z}_1$ during the intermediate linear regime depend on p and $\vec{q}$ in some complicated, nonlinear, and family-dependent way. Assuming that the dependence is analytic, we can determine the behavior to leading order from symmetry considerations. From the definition (10), and because $\vec{Z}_1$ carries the angular momentum during the intermediate linear regime, $\vec{B}$ is an odd function of $\vec{q}$, and A is an even function. Also, both A and $\vec{B}$ must vanish for $p = p_*$ and $\vec{q} = 0$ because by definition that initial data set is on the black hole threshold. To leading order in p and $\vec{q}$, we find that

$$Z(x, \tau) \simeq Z_*(x) + C_0(p - p_*) \, e^{\lambda_0 \tau} Z_0(x) + (C_1 \vec{q}) \cdot e^{\lambda_1 \tau} \vec{Z}_1(x, \Omega). \tag{11}$$

Here C_0 and C_1 are two unknown constants that depend on the two-parameter family we consider. C_1 is a 3×3 matrix. In axisymmetry it reduces to a number, but in general it is a non-trivial map between the vector $\vec{q}$ and the vector $\vec{Z}_1$, and therefore, in the end, between $\vec{q}$ and $\vec{a}$.

We now define τ_* such that

$$C_0 |p - p_*| e^{\lambda_0 \tau_*} \equiv \epsilon, \tag{12}$$

where ϵ is a fixed small constant. We then have

$$Z(x, \tau_*) \simeq Z_*(x) \pm \epsilon Z_0(x) + \vec{\delta} \cdot \vec{Z}_1(x, \Omega), \tag{13}$$

where

$$\vec{\delta} \equiv C_1 \vec{q} \left(\frac{\epsilon}{C_0 |p - p_*|} \right)^{\frac{\lambda_1}{\lambda_0}}. \tag{14}$$

Here the sign in front of ϵ is that of $p - p_*$. It appears because of the absolute value taken in the definition (12). Recall that in the following ϵ is a fixed positive constant, while $\vec{\delta}$ depends on the initial data.

In order to simplify our notation, it is convenient to introduce "reduced" values $\bar{p}$ and $\bar{\vec{q}}$ of p and $\vec{q}$ as

$$\bar{p} \equiv C_0 \epsilon^{-1} (p - p_*), \qquad \bar{\vec{q}} \equiv C_1 \vec{q}, \tag{15}$$

Note that by definition the direction of $\bar{\vec{q}}$ is now the direction of $\vec{a}$. In the following, this is implied, and we suppress the vector indices. In this simplified notation,

$$\delta = |\bar{p}|^{-\frac{\lambda_1}{\lambda_0}} \bar{q}, \tag{16}$$

The parameters $\bar{p}$ and $\bar{q}$ can be thought of as "reduced" values of p and $\vec{q}$, which hide the family-dependent part of the dependence of M and $\vec{a}$ on p and $\vec{q}$. They are similar to the "reduced temperature" and "reduced external field" in statistical mechanics. We can also think of them as coordinates on the phase space, in the way already discussed for $\bar{p}$.

We now consider $Z(x, \tau_*)$ as Cauchy data for a time evolution in τ that leads away from the perturbative regime and eventually to black hole formation or dispersion. $Z(x, \tau_*)$ contains the complete Cauchy data, up to an overall

scale, which must be provided separately. That scale is $e^{-\tau_*}$, which from our definitions is proportional to $|\bar{p}|^{-1/\lambda_0}$. In particular, the mass of the black hole that is formed must be proportional to this scale, with a constant of proportionality that can depend only on δ and the sign of $\bar{p}$. We find

$$M(p,q) \simeq |\bar{p}|^{\frac{1}{\lambda_0}} \begin{cases} F_M^+(\delta), & \bar{p} > 0 \\ F_M^-(\delta), & \bar{p} < 0 . \end{cases} \tag{17}$$

The functions $F_M^\pm(\delta)$ are universal, and so are the two exponents λ_0 and λ_1. Because of (10), F_M is an even function.

Consider the special case $\vec{q} = 0$, and hence $\delta = 0$. Then only the mode Z_0 is present in the intermediate linear regime, and, up to scale, we only need to consider the two data sets $Z_* - \epsilon Z_0$ and $Z_* + \epsilon Z_0$. It is known that the first data set disperses, while the second one forms a black hole. If we use $M = 0$ to denote the absence of a black hole, we have $F_M^-(0) = 0$ and $F_M^+(0) > 0$. We use the freedom to normalize $Z_0(x)$ in order to set $F_M^+(0) = 1$ as a convention. We then find

$$M(p,q) \simeq \begin{cases} \bar{p}^{\frac{1}{\lambda_0}}, & \bar{p} > 0 \\ 0, & \bar{p} < 0 \end{cases} \qquad \text{as } \bar{q} \to 0. \tag{18}$$

We have recovered the power law in its form (8), with the critical exponent $\gamma = 1/\lambda_0$ for the black hole mass.

We now consider the specific angular momentum $\vec{a}$ of the black hole. The dimensionless quantity a/M can depend on the initial data only through the dimensionless quantity δ, and on the sign of $\bar{p}$. M itself is given by (17), and we only have to put these two results together to obtain a. It is convenient to absorb the scaling function $F_M(\delta)$ into the new scaling function for a, and we can write

$$a(p,q) \simeq |\bar{p}|^{\frac{1}{\lambda_0}} \begin{cases} F_a^+(\delta), & \bar{p} > 0 \\ F_a^-(\delta), & \bar{p} < 0 \end{cases} \tag{19}$$

Here $F_a^\pm(\delta)$ are two new universal scaling functions. Clearly, $F_a^\pm(0) = 0$. Recall that we have adapted a simplified notation, in which F_a and δ are really vectors, but in which F_a takes its direction trivially from δ, so that F_a is really just a scalar function of a scalar variable. Because of (10), it is an odd function. If cosmic censorship holds, then $a < M$ must hold in black holes formed in gravitational collapse, and therefore $|F_a| < F_M$ for all δ.

To leading order F_a is proportional to δ. By adjusting the normalization of $\vec{Z}_1$, we can set the factor of proportionality to one. With this convention, and from the definition (16), we obtain a power law for a in the limit of small $\bar{q}$:

$$a(p,q) \simeq \begin{cases} \bar{q}\, \bar{p}^{\frac{1-\lambda_1}{\lambda_0}}, & \bar{p} > 0 \\ 0, & \bar{p} < 0 \end{cases} \qquad \text{as } \bar{q} \to 0. \tag{20}$$

This had been obtained previously under the assumption that $\lambda_1 < 0$ [10].

VI. PROPOSED NUMERICAL TESTS

So far our results have been formal: we know the two exponents λ_0 and λ_1, but not the universal scaling functions. However, we predict that such functions exist and are universal. This can be tested by comparing two or more two-parameter families of initial data. The two family-dependent parameters C_0/ϵ and C_1 (the latter is a matrix) must be determined for each family, by fitting observations in the limit $q \to 0$ to the formulas (18) and (20). The scaling functions can then be read off from the first family, and tested against the second and any further family.

Alternatively, the scaling functions can be calculated directly by evolving the data (13) for fixed ϵ and all values of δ. In the notation of [11], the area radius r in spherical symmetry is related to the dimensionless radial coordinate x and the scale coordinate and coordinate τ by $r \equiv sxe^{-\tau}$. (s is a known constant whose significance does not matter here.) We set $\tau = 0$, and consider the initial data

$$Z_{\text{initial}}(r, \Omega) = Z_*(r/s) \pm AZ_0(r/s) + BZ_1(r/s, \Omega). \tag{21}$$

Here $Z_*(x)$ is already known, and $Z_0(x)$ and $Z_1(x)$ are known up to normalization [11]. From the three-fold degenerate $\vec{Z}_1$, we arbitrarily choose $m = 0$. The actual dimensionless variables that Z stands for in perfect fluid collapse are defined in [11]. The physical, dimensionful fluid variables of [15] are obtained from the rescaled variables defined in [11] by setting $\tau = 0$ everywhere.

We now evolve these data nonlinearly in axisymmetry. In order to explore all values of δ in the universal scaling functions, we could keep A fixed at a small value $\pm\epsilon$, and vary B from 0 to ∞. In practice, however, we want to vary the ratio A/B from $-\infty$ to ∞, but choose an overall factor in both A and B small enough so that the initial data are in the intermediate linear regime but large enough so that the evolution leaves this regime soon. This minimizes the range of scales that the nonlinear evolution code has to cover, and may make the calculation possible without the need for adaptive mesh refinement.

Clearly, $\bar{p}$ is proportional to A and $\bar{q}$ is proportional to B. The constants of proportionality, however, are unknown because we do not have the correct normalizations of Z_0 and Z_1. We just fix an arbitrary normalization, and put in two adjustable constants. Then the mass and specific angular momentum of the final black hole that is created from the initial data (21) are related to the scaling functions by

$$M(\pm\alpha A, \beta B) = A^{\frac{1}{\lambda_0}} F_M^{\pm}\left(BA^{-\frac{\lambda_1}{\lambda_0}}\right), \tag{22}$$

$$a(\pm\alpha A, \beta B) = A^{\frac{1}{\lambda_0}} F_a^{\pm}\left(BA^{-\frac{\lambda_1}{\lambda_0}}\right). \tag{23}$$

The constants α and β correspond to the unknown correct normalizations of Z_0 and Z_1, and must be adjusted to obtain the two conventions

$$F_M^{+}(0) = 1, \qquad F_a^{+}{}'(0) = 1. \tag{24}$$

VII. LARGE ANGULAR MOMENTUM

Current axisymmetric rotating fluid evolution codes are not ready at the time of writing to calculate the universal scaling functions in the manner suggested in the previous section. It is therefore tempting to speculate about the form of the universal scaling functions at large δ.

Consider the limit where $\bar{p} \to 0$ while $\bar{q} \neq 0$ so that $\delta \to \infty$. In physical terms this corresponds to first fine-tuning a one-parameter family of initial data without angular momentum to the black hole threshold and then adding a small but finite amount of angular momentum to the data. Do the data obtained in this way form a black hole? If our model is correct, the answer is universally yes or no, independently of the family of zero angular momentum data we have fine-tuned and independently of how we have added that bit of angular momentum. The reason is that such data are funneled through the data set $Z_* + \varepsilon\vec{Z}_1$ (for some small fixed ε) in the intermediate linear regime. We now discuss the two possible final states for these data in turn.

Possibility 1: The data $Z_* + \varepsilon\vec{Z}_1$ form a black hole. As its mass is finite (these data possess an intrinsic scale), the explicit power of $\bar{p}$ in formula (17) must be canceled by a power-law behavior of $F_M(\delta)$ as $\delta \to \infty$. One easily sees that this power law must be

$$F_M^{\pm}(\delta) \simeq C_M|\delta|^{\frac{1}{\lambda_1}}, \quad |\delta| \to \infty \tag{25}$$

for the cancellation to occur. But then one immediately obtains that

$$M \simeq C_M|\bar{q}|^{\frac{1}{\lambda_1}}, \qquad |\delta| \to \infty. \tag{26}$$

By the same argument we have

$$F_a^{\pm}(\delta) \simeq C_a \, \text{sign}(\delta) \, |\delta|^{\frac{1}{\lambda_1}}, \quad |\delta| \to \infty, \tag{27}$$

and therefore

$$a \simeq C_a \, \text{sign}(\bar{q}) \, |\bar{q}|^{\frac{1}{\lambda_1}}, \qquad |\delta| \to \infty. \tag{28}$$

The constants C_a and C_M are again universal. We see that a/M goes to the constant C_a/C_M as we increase the angular momentum in the initial data. We do not know what the value of this constant is, except that cosmic censorship requires $C_a/C_M \leq 1$.

Possibility 2: The data $Z_* + \varepsilon\vec{Z}_1$ do not form a black hole. On physical grounds this alternative appears more likely, as one would expect centrifugal forces to disrupt data that already hover between collapse and dispersion. This means that $F_{M,a}^{\pm}(\pm\infty) = 0$. If we consider obtaining this limit by adding angular momentum to supercritical

data, that is by increasing $\bar{q}$ at fixed $\bar{p} > 0$, it appears likely that above a threshold amount of angular momentum no black hole is formed at all, rather than ever smaller ones. In this case it also seems plausible that zero angular momentum data below the black hole threshold will not form a black hole if one adds angular momentum. If these two assumptions are true, then instead of (25) and (27) we have

$$F_{M,a}^+(\delta) = 0 \quad \text{for } |\delta| > \delta_{\max}, \quad F_{M,a}^-(\delta) = 0 \quad \text{for all } \delta. \tag{29}$$

This means that black holes are formed if and only if

$$\bar{p} > 0 \quad \text{and} \quad |\bar{q}| < \delta_{\max} \, \bar{p}^{\frac{\lambda_1}{\lambda_0}}. \tag{30}$$

This defines a convex region in the p, q plane. We do not know if black holes at the threshold with non-zero angular momentum are formed with zero or finite mass. This gives rise to two sub-cases:

Possibility 2a: In the presence of angular momentum there is a mass gap at the black hole threshold. This means that F_M^+ is discontinuous at $\delta_{\max}$, of the form

$$F_M^+(\delta) \simeq \begin{cases} K_M, & \delta \lesssim \delta_{\max} \\ 0, & \delta > \delta_{\max} \end{cases} \tag{31}$$

for some universal constant K_M and the size of the mass gap is

$$\Delta M \simeq K_M \, \bar{p}^{\frac{1}{\lambda_0}} \tag{32}$$

Similar behavior would hold for F_a^+, for some universal constant K_a so that

$$\Delta a \simeq K_a \, \bar{p}^{\frac{1}{\lambda_0}} \tag{33}$$

Therefore, the spin/mass ratio of black holes formed at the threshold is universal, with

$$\frac{\Delta a}{\Delta M} = \frac{K_a}{K_M}. \tag{34}$$

From cosmic censorship we have $K_a/K_M < 1$. Fig. 1 gives a qualitative impression of $M(\bar{p}, \bar{q})$ and $a(\bar{p}, \bar{q})$ in case 2a.

Possibility 2b: There is no mass gap even in the presence of angular momentum. This means that $F_M^+(\delta)$ vanishes at $\delta_{\max}$. If this happens, for example, as a power n of distance from the threshold,

$$F_M^+(\delta) \simeq \begin{cases} K_M(\delta_{\max} - \delta)^n, & \delta \lesssim \delta_{\max} \\ 0, & \delta > \delta_{\max} \end{cases} \tag{35}$$

then the black hole mass would scale as that same power n of any regular coordinate scalar on the phase space that vanishes at the threshold (except at $\vec{q} = 0$). Fig. 2 gives a qualitative impression of $M(\bar{p}, \bar{q})$ and $a(\bar{p}, \bar{q})$ in case 2b.

VIII. THE ANALOGY WITH STATISTICAL MECHANICS

This work was motivated by the attempt to exploit the critical phase transition/critical collapse analogy to learn something new about critical collapse. The preceding material has been presented in a self-contained manner without any explicit reference to statistical physics concepts, but it may be interesting to point out the exact parallels now.

The calculation of critical exponents in critical collapse is mathematically identical to the calculation of critical exponents in statistical mechanics, even though the underlying physical phenomena are totally different. The mathematical equivalent of the renormalization group acting on micro-states in statistical mechanics is the time evolution, in certain preferred coordinates, acting on initial data in general relativity. Both can be considered as dynamical systems. Critical exponents are calculated by linearizing around their critical fixed points [2,16].

In statistical mechanics, the critical fixed point typically has two growing modes. Physically these are linked to the temperature T and to a generalized external force. It is helpful to consider two contrasting examples. For a fluid confined in a vessel so that a vapor phase is in equilibrium with a liquid phase, the generalized force is the pressure P, and the order parameter is the difference $\rho_{\text{liquid}} - \rho_{\text{gas}}$ between the densities of the two phases. For a ferromagnet, the generalized force is the external magnetic field $\vec{H}$, and the order parameter is the magnetization $\vec{m}$. In both cases the temperature-force plane contains a line of first-order phase transitions ending in a second-order

phase transitions. (In the standard terminology, a first-order transition is one where the order parameter jumps from zero to a non-zero value. A second-order, or critical, phase transition is one where the order parameter is continuous, rising as a power-law.) For the fluid, this is the liquid-gas phase transition, ending at the critical point where these two phases have the same density. For the ferromagnet, the phase transition is between the possible directions of the magnetization $\vec{m}$ at $T < T_C$. This first-order transition ends at the Curie temperature T_C. The spontaneous magnetization at zero external field behaves as a power of $T_C - T$ for $T \lesssim T_C$.

If one restricts consideration to a vanishing external magnetic field, one appears to have an analog of the black-hole threshold in spherical symmetry, where the $T_C - T$ corresponds to $p - p_*$ and the absolute value of the magnetization $|\vec{m}|$ corresponds to the black hole mass M. The formula (1) corresponds to the dependence of the spontaneous magnetization on the temperature, in the absence of an external field. But typical critical phase transitions in statistical mechanics have two independent critical exponents, corresponding to two growing modes of the fixed point. Nigel Goldenfeld has asked if a second growing mode could not be found in the critical collapse problem so that the universal critical behavior would show not one critical exponent but two and, more interestingly, universal scaling functions [17].

Here we suggest that the analog of the external magnetic field $\vec{H}$ in the collapse problem is the angular momentum parameter $\vec{q}$. Angular momentum provides a second growing mode of the critical point. The analog of the order parameter $\vec{m}$, a vector, is not the black hole mass but its (specific) angular momentum vector $\vec{a}$. Conversely, the analog of the black hole mass M is a scalar that scales like a length, for example the correlation length ξ. (The fact that ξ diverges at the critical point, while M goes to zero because that is what length scales do in the two kinds of critical point.) We have shown that a second critical exponent and universal scaling functions do indeed arise, in exact mathematical analogy with the ferromagnet model [16].

In both the ferromagnet and collapse models the two growing modes have different symmetries. While $Z_* + \epsilon Z_0$ forms a black hole, $Z_* - \epsilon Z_0$ disperses. The two signs are qualitatively different. But $Z_* + \epsilon Z_0 + \vec{\delta} \cdot \vec{Z}_1$ will either collapse or disperse depending only on the absolute value of δ, independently of its direction. Similarly in the ferromagnet, the state at $T = T_C + \epsilon$ differs qualitatively from that at $T = T_C - \epsilon$, but only the absolute value of $\vec{H}$ matters. In each case, the vector-valued parameter ($\vec{q}$ or $\vec{H}$) is associated with a symmetry breaking. Because $\vec{q} = 0$ or $\vec{H} = 0$ correspond to an unbroken symmetry, their critical value is obviously zero, whereas the critical values T_C and p_* are nontrivial. (T_C depends on the material, and p_* on the family.) If one imposes the symmetry on the problem (no external magnetic field, or octant symmetry in collapse), the critical point has effectively only one unstable mode.

There is one major difference between the ferromagnet and critical collapse: In the ferromagnet, an infinitesimal external field at a temperature slightly above the Curie temperature creates a finite magnetization. The ferromagnetic region is therefore concave in the T-H plane. Even with zero external field, the symmetry is spontaneously broken, and a finite net magnetization in a random direction results. Fig. 3 gives a qualitative impression of $m(T_C - T, H)$ for a ferromagnet in axisymmetry, and should be contrasted with Figs. 1 and 2. A partial analog of this in critical collapse would be possibility 1 in the previous section. However, possibility 2 appears more likely on the physical grounds that centrifugal forces should oppose collapse. The black hole region is then convex in the $\bar{p}\bar{q}$ plane.

The analogy between critical collapse and the critical point of a fluid is less close than between critical collapse and the ferromagnet, as neither temperature nor pressure are associated with a symmetry breaking: both have to be fine-tuned at once to nontrivial critical values that depend on the material. As a curiosity, however, we note that an analog of the liquid-gas transition, where q is a scalar parameter, and not associated with a symmetry breaking, could arise in critical collapse in the context of semiclassical gravity. Self-similarity in critical collapse in semiclassical gravity is broken at small scales (of the order of the Planck scale) by the quantum stress-energy. Brady and Ottewill [18] suggest that the effect of the quantum stress-energy can be described approximately in terms of a single unstable mode (in addition to the known classical one) with a growth exponent of $\lambda_1 = 2$. If this description is correct, the parameter q would describe some measure of the excitation state of the quantum fields.

IX. CONCLUSIONS

The black hole threshold in gravitational collapse has been investigated until now only for initial data without angular momentum. It is largely unknown what happens when the threshold is crossed along a family of initial data with nonzero angular momentum. If the angular momentum is very small, it can be treated as a perturbation of spherical collapse throughout, and for this case a critical exponent for the black hole angular momentum has been predicted [10–12] and is now waiting to be tested in numerical collapse simulations.

Here we have noted that the perturbative calculations giving rise to this prediction can be extended in the case where the angular momentum perturbation constitutes a second growing mode of the self-similar, spherically symmetric critical solution, in addition to the spherical growing mode known already. We can now extend our predictions to

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a regime where the angular momentum of the initial data is small, but large enough so that the final black hole is rapidly rotating.

In Eqs. (17) and (19) we have made predictions for the black hole mass and specific angular momentum in this regime, expressed in terms of two known critical exponents and two yet unknown functions of one variable δ. The prediction that universal scaling functions exist at all can be tested by comparing two or more two-parameter families of initial data. The scaling functions could also be calculated more simply and precisely through the nonlinear evolution of only two one-parameter families of axisymmetric initial.

The necessary initial data are already known to high precision, and the necessary technology (an axisymmetric fluid code that can form rotating black holes, but without mesh refinement) should be available soon. In the meantime we have discussed the qualitative features of these functions, distinguishing two possibilities, and suggesting one of them as the most plausible on physical grounds.

The prediction of universal scaling functions in critical collapse with angular momentum both extends and clarifies the analogy between critical phenomena in gravitational collapse and critical phase transitions in statistical mechanics, in particular the ferromagnetic phase transition.

ACKNOWLEDGMENTS

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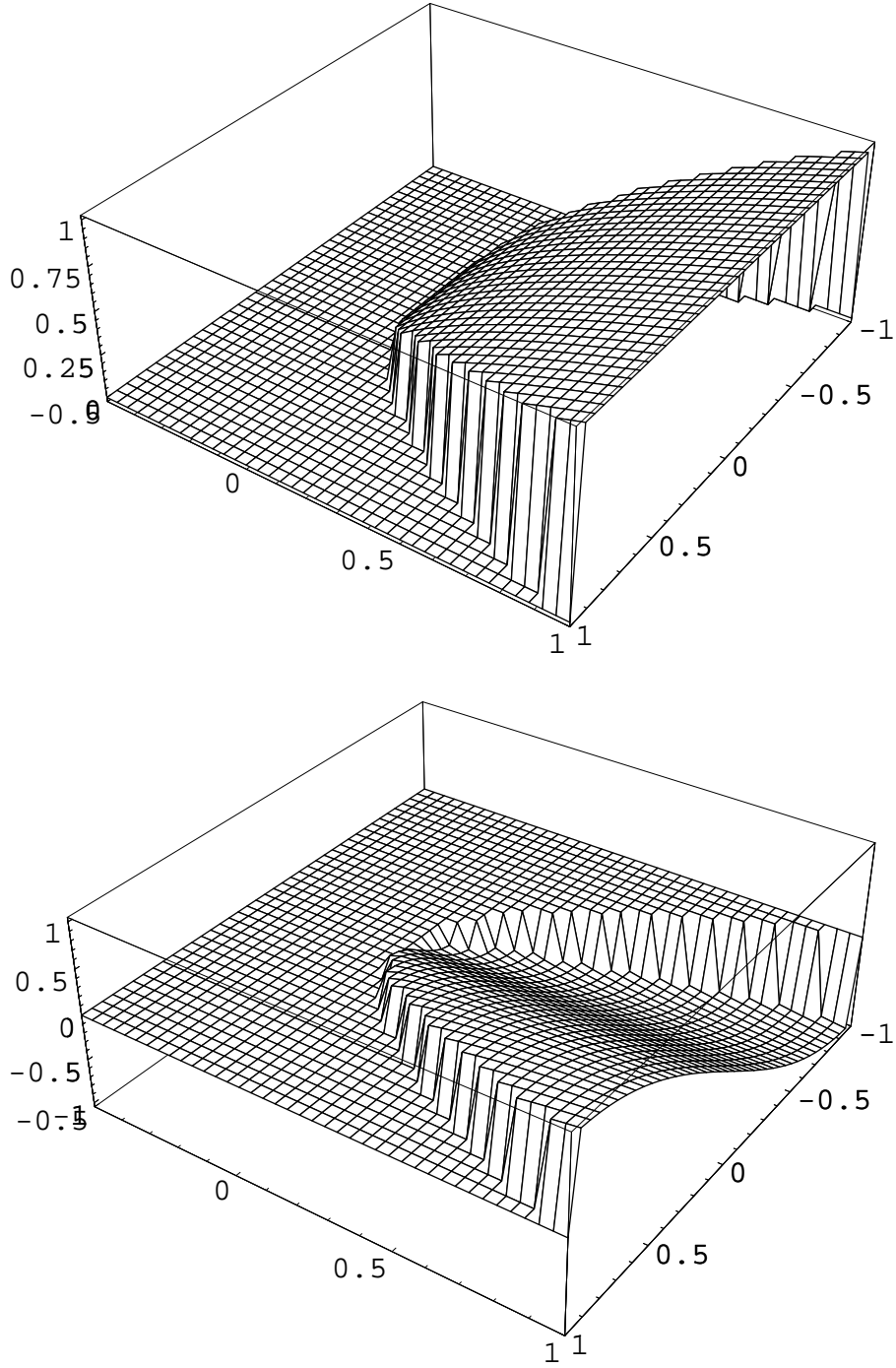[19] P. Bizon, T. Chmaj and Z. Tabor, Phys. Rev. D **59**, 104003 (1999).

FIG. 1. Schematic plot of black hole mass M and specific angular momentum a in axisymmetry as functions of $\bar{p}$ and $\bar{q}$, assuming case 2a. For illustration we have assumed $\lambda_0 = 2$, $\lambda_1 = 1$, $F_M^+(\delta) = \theta(1 - |\delta|)$ and $F_a^+(\delta) = \theta(1 - |\delta|)\sin(\pi\delta/2)$. Note that in reality $\lambda_1/\lambda_0 \ll 1$, and that we have assumed that $|a|/M^2 \to 1$ at the black hole threshold.
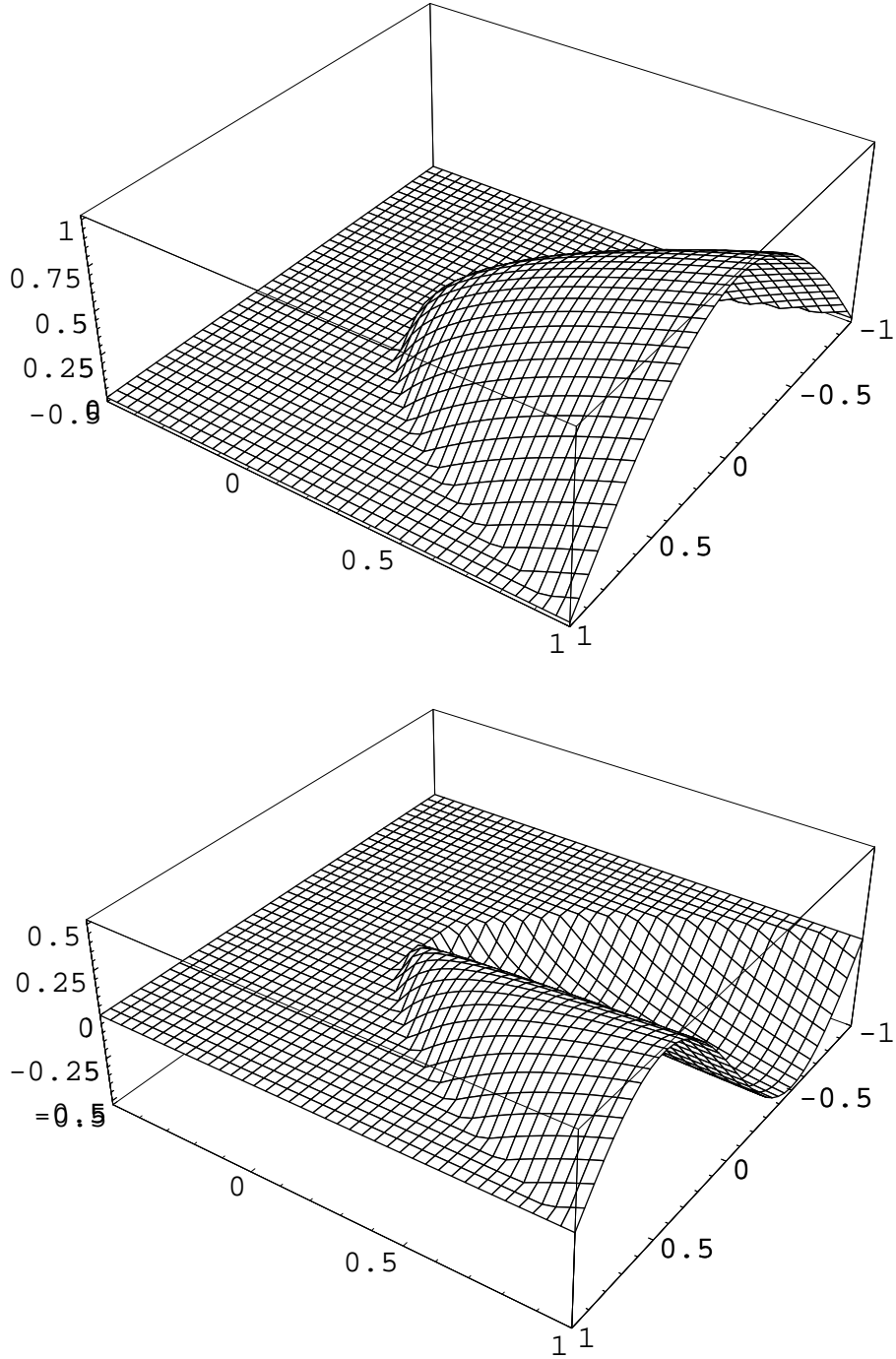
FIG. 2. Schematic plot of black hole mass M and specific angular momentum a in axisymmetry as functions of $\bar{p}$ and $\bar{q}$, assuming case 2b. For illustration we have assumed $\lambda_0 = 2$, $\lambda_1 = 1$, $F_M^+(\delta) = \theta(1-|\delta|)\cos(\pi\delta/2)$ and $F_a^+(\delta) = \theta(1-|\delta|)\sin(\pi\delta)/2$. Note that in reality $\lambda_1/\lambda_0 \ll 1$, and that we have assumed that $|a|/M^2 \to 1$ at the black hole threshold.
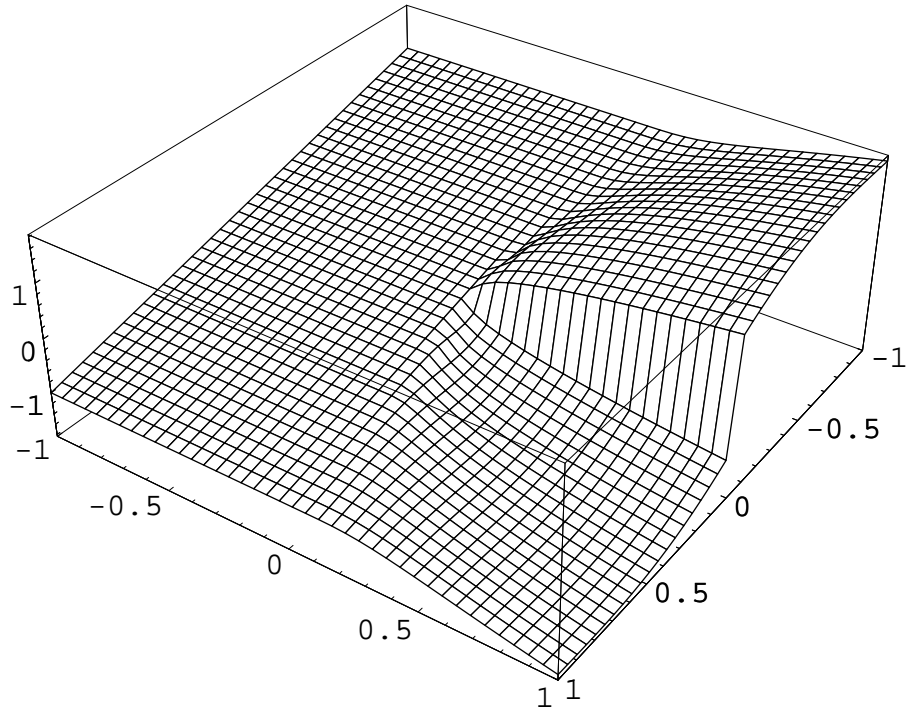
FIG. 3.   Schematic plot of the magnetization m of a ferromagnet as a function of $T_C - T$ and H. For illustration we have assumed $\lambda_0 = 2$, $\lambda_1 = 1$, $F^-(\delta) = \text{sign}(\delta)|\delta|^{1/\lambda_1}$ and $F^+(\delta) = F^-(\delta) + e^{-\delta^2}[\text{sign}(\delta) + \delta]$.